Combinatorics Olympiad
Advanced & Free Levels
November 2024

Problems

1. Consider a 13×13 table in which the rows are numbered from top to bottom and the columns are numbered from left to right by numbers 1 to 13. Assume that the cells that are intersections of an even row with an even column are blue and the other cells are red (There are 36 blue cells in the table). What is the maximum number of 1×3 and 3×1 tiles that we can place inside this table such that (1) each rectangle only contains red cells and (2) no cell belongs to more than one of these rectangles?

2. We are given a permutation of $\{1, 2, ..., n\}$, say $\Pi =< \pi_1, ..., \pi_n >$. The swapping step is defined as follows:

   - Choose two numbers s and t, such that $1 \leq s < t \leq n$ and $t - s$ is odd.
   - For each $0 \leq i < \frac{t-s+1}{2}$, update Π by swapping π_{s+2i} and π_{s+2i+1}.

   For instance, employing one swapping step with $s = 3$ and $t = 8$ on the permutation $< 10, 9, 8, 7, 6, 5, 4, 3, 2, 1 >$ results in the permutation $< 10, 9, 7, 8, 5, 6, 3, 4, 2, 1 >$.

   Assume that $n \geq 3$ and we start with $\Pi =< n, n-1, \ldots, 1 >$.

   a) Prove that at least n steps is required to modify Π to $< 1, 2, ..., n >$.
   b) Prove that it is possible to modify Π to $< 1, 2, ..., n >$ in exactly n steps.

3. Let $A_1, \ldots, A_{20}$ be 20 distinct subsets of size three of the set $X = \{1, 2, \ldots, 10\}$. We say that a subset S of X is a covering subset if for every $1 \leq i \leq 20$, it holds that $S \cap A_i \neq \emptyset$. What is the minimum possible value of k, such that there always exists a covering subset of size k?

4. We call a graph *planar* if it can be drawn on the plane in such a way that its edges intersect only at their endpoints. Let $n \geq 3$ be an integer and let G be a planar graph on n vertices. Determine the maximum possible number of cycles of length 3 of G in terms of n.

5. Let G be a simple graph and let V be the set of vertices of G. We denote by $f(G)$ the maximum number k such that there exists a subset $S \subseteq V$ with $|S| = k$, in which every vertex in S has at most one neighbor in S.

   a) Compute $f(G)$ if $V = \{0, 1, \ldots, n-1\}$ and $E = \{uv | u - v \equiv r \pmod{n} \text{ where } r \in \{-2, -1, 1, 2\}\}$.
   b) Assume each vertex $v \in V$ corresponds to a unique sequence of length n consisting of elements in $\{0, 1, 2\}$ ($|V(G)| = 3^n$). A pair of vertices are adjacent in G if their corresponding sequences differ in exactly one position. for example if $n = 4$ vertices corresponding to sequences $< 0, 1, 2, 0 >$ and $< 0, 1, 1, 0 >$ only differ in the third position so they are adjacent.Prove that $f(G) > 3^{n-1}$.

6. Consider the increasing sequence $1, 2, \ldots, n$. In each move, we first take two adjacent elements x and y that are still positive and then by spending $\min\{x, y\}$ gold coins we decrease both elements by $\min\{x, y\}$. What is the minimum number of gold coins that we have to spend in order to reach a sequence in which no more moves are possible?

7. There are 100 points on the plane, namely $P_1, \ldots, P_{100}$ such that no three points are collinear. Assume for every i and j such that $1 \leq i < j-1 \leq 98$, the segments P_iP_{i+1}, P_jP_{j+1} do not intersect (i.e. do not share a common point). We say that $\{P_i, P_{i+1}, P_{i+2}\}$ is an *empty* consecutive triple if the triangle $\triangle P_iP_{i+1}P_{i+2}$ does not contain any of the other points. Find the largest number k such that one can always find at least k empty consecutive triples.