Combinatorics Olympiad
Advanced & Free Levels
October 2024

Problems

1. Morteza and Joseph have 100 boxes such that for each $1 \leq i \leq 100$ one of these boxes has exactly i coins. Joseph puts the boxes on top of each other in an order that he prefers. Now in 10 steps Morteza collects coins in the following way:

    > at each step Morteza picks up the top 10 boxes, opens them and then selects one of the boxes that he has already opened (this also includes all the boxes that he has openened in the previous steps) and collects all the coins of that box.

    What is the maximum number of coins that Morteza can always collect regardless of the way Joseph orders the boxes?

2. Let S be a set of six points in the plane such that no three points of S are collinear. In how many ways can these 6 points be partitioned into two groups of three such that the sides of the triangle formed by the points in each group do not intersect the sides of the triangle formed by the other group?

3. We say that a sequence $x_1, x_2, \ldots, x_n$ is increasing if $x_i \leq x_{i+1}$ for all $1 \leq i < n$. How many ways are there to fill an 8×8 table by numbers 1, 2, 3, and 4 such that:

    - The numbers in each row are increasing from left to right,
    - The numbers in each column are increasing from top to bottom,
    - and there is no pair of adjacent cells such that one is filled with 2 and the other one is filled with 3. (We say two distinct cells are adjacent if they share a side)

4. Matin and Morteza are playing a game together on a graph with 100 vertices. At the beginning of the game, the graph has no edges. In each turn, Matin picks a vertex that is not *full*, say u and Morteza must add a new edge that has u as one of its endpoints while keeping the graph simple (A vertex is full if it is adjacent to all other vertices of the graph). The game ends as soon as the graph has a cycle of even length. Matin wishes to maximize the number of edges, whereas Morteza wants to minimize this. Assuming both players play optimally, what is the number of edges of the final graph?

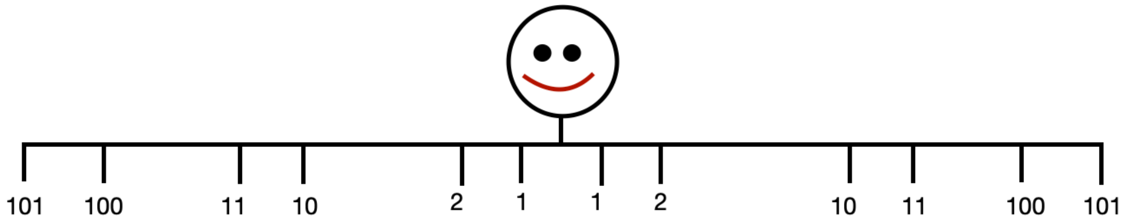
5. We say that a coloring of the 7×7 table is nice if,

    - Every cell is colored by blue or red,
    - Every cell has precisely one diagonal neighbor that is red.

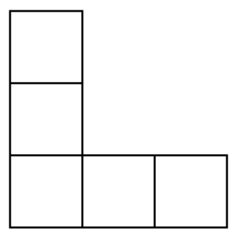
    Determine the number of nice colorings.

    (A cell A is a diagonal neighbor of cell B if A and B share exactly one point. For instance, each of the four cells on the corners of the table has one diagonal neighbor.)

6. Below we have a figure of a spider named Johny who has 12 feet. The numbers in the figure correspond to the distance of Johny's feet from his head. For taking a nap on his web, Johny doesn't need to hold all his feet down but he must keep his balance and that means, (1) he should hold at least one foot down on the web, and (2) the total distances of his left feet that are on the web from his head should be equal to the total distances of his right feet that are on the web from his head. Determine the number of ways that Johny can take a nap.



7. In a 7×7 table initially all the cells are white. At each step we can identify a tile as depicted in the figure below (with rotations allowed) on the table and flip the colors (from white to black and black to white) of all those five cells in the table simultaneously. After finitely many steps, what is the maximum possible number of black cells in the table?



8. Given a sequence $S =< a_1, \ldots, a_n >$ of integers, the number of inversions of S is equal to the number of pairs $1 \leq i < j \leq n$ such that $a_i > a_j$. Let $Z =< a_1, \ldots, a_{20} >$ be a sequence of 20 elements from $\{1, \ldots, 10\}$, where $a_1, \ldots, a_{10}$ is a permutation of $\{1, \ldots, 10\}$ and for every $1 \leq i \leq 10$, $a_{10+i} = 11 - a_i$. Let A and B be the minimum and maximum possible values for the inversion number of Z, respectively. What is $A+B$?

9. Rostam wants to find a sequence of numbers $< a_1, ..., a_{2025} >$ such that $0 \leq a_i \leq 1023$ and if you place them around a circle in the same order (as they appear in the sequence), then each number is **XOR** of its two neighbors on the circle. How many ways are there to do this?

   The XOR of two non-negative integers x and y is defined in the following way: Assume $(\overline{x_k ... x_1})_2$ and $(\overline{y_t ... y_1})_2$ are their corresponding binary representations and without loss of generality, assume $k \geq t$. Now, let $y_{t+1} = \cdots = y_k = 0$.

   Then, the binary representation of their XOR is $(\overline{z_1 ... z_k})_2$ where $z_i = 0$ if $x_i = y_i$ and $z_i = 1$ if $x_i \neq y_i$.

   For instance, let $x = 49$ and $y = 101$, then $x = (110001)_2$ and $y = (1100101)$. Now, the binary representation of their XOR is $(1010100)_2$ which means their XOR is 84.

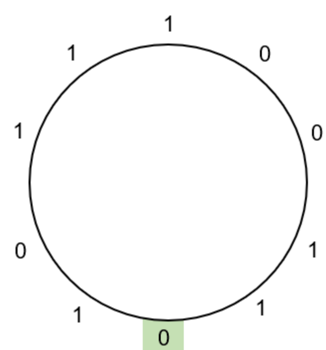
10. Bahman has an 8×8 table. Originally, all the cells are empty and white. He fills each of the cells with a number from 1 to 4. Then, he chooses two numbers a and b such that $1 \leq a < b \leq 4$ and colors all the cells that are filled with a or b in gray. The value of the table is $64 \times t$, where t is the number of rows that are completely gray after this process. For instance, in the table below, if we choose $i = 2$ and $j = 3$, we would have $t = 3$ rows that are completely gray and hence the value of the table is 192. What is the average value of the table over all the possible ways that Bahman can fill the table and choose a and b?

1	2	1	3	2	3	3	4
1	3	3	3	2	1	2	1
4	2	4	1	4	3	4	4
2	2	2	3	3	2	3	2
1	2	3	4	1	2	3	4
2	2	2	2	2	2	2	2
3	2	3	3	3	2	3	2
3	4	4	2	2	1	1	1

11. There are 10 cells around a circle and in each cell there is a digit which is either one or zero. The process of creating a string from a given cell C is to start from C and an empty string S and then in 10 steps, we repeat the following: Move to one of the two neighboring cells and append the digit inside that cell to the end (right) of S.

    Assume from <u>all the cells</u> we are able to create $S = 0101010101$. How many different ways are there to initialize the values of all the cells with zero or one such that this is possible?

    For instance, in the circle below, if we start from the colored cell, we cannot create $S = 0101010101$ but we can create $S = 1010101010$. Two ways of initializing the values of the cells such as A and B are the same if you can reach A by shifting B around the circle.



12. Assume S_n is the set of all ordered n-tuples of 0 and 1 and let $A_1, A_2, \ldots, A_{32}$ be a permutation of the elements of S_5. Also assume that $f(A_1) = 1$ and for every $1 \leq i \leq 32$ the value of $f(A_i)$ is equal to the smallest positive integer such that for every j $(1 \leq j < i)$, where A_i and A_j differ in exactly one coordinate, it holds that $f(A_i) \neq f(A_j)$ (For instance, if $A_i = (0, 1, 1, 0, 1)$ and $A_j = (0, 1, 0, 0, 1)$ then A_i and A_j differ only in the third coordinate). What is the maximum possible value of $\max\{f(A_1), f(A_2), \ldots, f(A_{32})\}$.

Switching

A graph is a *tree* if it is connected and does not contain any cycles. Let T be a tree. The *diameter* of T is defined as the length (the number of edges) of the longest path of T. At each step we are allowed to update T by removing an edge and then drawing a new edge such that T remains a tree.

13. If T has 2024 vertices and is of diameter 100, after one step what is the minimum possible diameter of T?

14. If T has 2024 vertices and is of diameter 100, after one step what is the maximum possible diameter of T?

15. What is the minimum possible number k, such that for every tree T with 2024 vertices and diameter 100, one can reach a tree of diameter smaller than 100 in at most k steps.